Proximal Bundle Method for simplified unilateral adhesion contact problem of elasticity

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Abstract. We consider a mathematical model which describes the adhesive contact between a linearly elastic body and an obstacle. The process is static and frictionless. The normal contact is governed by two laws. The first one is Signorini law, representing the fact that there is no penetration between body and obstacle. The second one is a Winkler type law signifying that if there is no contact, the bonding force is proportional to the displacement below a given bonding threshold and equal to zero above the bounding threshold. The model leads to variational-hemivariational inequality. We present the numerical results for solving simple two-dimensional model problem with Proximal Bundle Method (PBM). We analyze the method sensitivity and convergence speed with respect to its parameters.

Keywords: linearly elastic body, Winkler law, unilateral contact, adhesion, variational-hemivariational inequality, proximal bundle method

1. Introduction

The laws which describe contact between an elastic body and a foundation often are discontinuous, nonmonotone and multivalued. Such laws lead to problems which can be represented by means of hemivariational inequalities (HVIs) which have been introduced by Panagiotopoulos in 80’s (see for example [17]). The requirement for the contact law to be possible to describe by means of HVI is the existence of a locally Lipschitz superpotential such that the multivalued law corresponds to its Clarke subdifferential. This is not possible for nonpenetration Signorini law for which
the (convex) superpotential assumes infinite values. Therefore solving the contact problems which involve both nonmonontonicity and infinite values (which is the case in nonpenetration problems) requires the so called variational - hemivariational inequalities.

Hemivariational inequalities describing various problems of contact mechanics were investigated for example in [17] and, later, in [15]. Variational - hemivariational inequalities in context of contact mechanics were studied, among others, by Panagiotopolos [18], Motreanu [16], Liu [11], [12], Carl [2] and Kovtunenko [8], [10].

This article deals with the numerical solution of variational hemivariational inequalities. We consider a static problem which represents a simplified model of unilateral adhesive contact between a linear elastic body and a foundation. The adhesive law similar to the one used in this article (see formulae (8)-(9) in the sequel) was used in [7] (Example 6.4) and [6] (Application 6.6.8).

After discretization using the Finite Element Method (FEM), variational-hemivariational inequality can be represented as the nonconvex optimization problem [7]. This problem can be solved numerically using the nonsmooth optimization methods like Proximal Bundle Method or Bundle Newton Method. Such approach is used in [7] (see also [13], [14]). The Proximal Bundle Method (PBM) has parameters which are to be chosen arbitrarily (Makela [13], [14] describes the method of the adaptive choice of one parameter $u_k$ which is key for convergence rate). The main aim of this article is to investigate the sensitivity of the PBM applied to variational - hemivariational inequality describing the adhesive unilateral contact with respect to the those parameters. The convergence rate for the PBM is known to be very slow: for the convex potential one requires $O(1/\epsilon^3)$ iterations to obtain the error of order $\epsilon$ (see [9]). Here we present the numerical convergence rate analysis for the nonconvex problem originating from the adhesive contact model.

The example problem used for computations is similar to the delamination problem used as a benchmark in [7] (Example 6.4), however, the structured finite element mesh used in this article consists of the rectangles with two diagonals (so called crossed mesh) and such mesh in known to avoid the numerical locking effects in elasticity problems in contrast to the mesh with one diagonal [1]. For the benchmark problem the detailed analysis of the PBM convergence with respect to the parameter choice is presented.

The structure of the article is the following. After the Section 2. in which the required definitions are introduced, we present the abstract problem in the Section 3. In Section 4. the PBM method is recalled. Section 5. presents the benchmark problem used for computations. Section 6. is divided into few parts presenting sensitivity tests and conclusions.

2. Notation and definitions

Let $\Omega \subset \mathbb{R}^d$, where $d \in \{2, 3\}$ be nonempty, open and bounded set with regular (Lipschitz) boundary. This set is occupied by the linearly elastic body. The boundary of
\( \Omega \) is separated into three pairwise disjoint subsets \( \partial \Omega = \Gamma_C \cup \Gamma_D \cup \Gamma_N \). We denote

\[
H = L^2(\Omega)^d = \{ u = (u_i) | u_i \in L^2(\Omega) \},
\]

\[
Q = \{ \sigma = (\sigma_{ij}) | \sigma_{ij} = \sigma_{ji} \in Q \},
\]

\[
H_1 = \{ u = (u_i) | \varepsilon(u) \in Q \},
\]

\[
Q_1 = \{ \sigma \in Q | \text{Div} \sigma \in H \}.
\]

Here \( \varepsilon : H_1 \rightarrow Q \) and \( \text{Div} : Q_1 \rightarrow H \) are the deformation and divergence operators, respectively, defined by

\[
\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}),
\]

\[
\text{Div} \sigma = (\sigma_{ij,j}),
\]

where \( i \) and \( j \) run between 1 and \( d \), and the summation convention over repeated indices is adopted. Index after comma denotes partial derivative with respect to the corresponding component of the independent variable.

We denote the normal and tangential components of \( \sigma \) by \( \sigma_v \) and \( \sigma_t \). If \( \sigma \) is smooth enough then

\[
\sigma_v = (\sigma n) \cdot n, \quad \sigma_t = \sigma n - \sigma_v n.
\]

**Definition 1.** The Clarke generalized directional derivative (see [3]) of a locally Lipschitz function \( h : X \rightarrow \mathbb{R} \) at the point \( x \in X \) in the direction \( v \in X \), where \( X \) is a reflexive Banach space, denoted by \( h^0(x; v) \), is defined by

\[
h^0(x; v) = \limsup_{y \rightarrow x, \lambda \downarrow 0} \frac{h(y + \lambda v) - h(y)}{\lambda}.
\]

The Clarke subdifferential of \( h \) at \( x \) denoted by \( \partial Cl h(x) \) is a subset of \( X^* \) given by

\[
\partial Cl h(x) = \{ \xi \in X^* : h^0(x; v) \geq \langle \xi, v \rangle_{X^* \times X} \text{ for all } v \in X \}.
\]

The subdifferential of a convex functional will be denoted by \( \partial \text{Conv} \).

### 3. Problem formulation

In this section we present a contact problem of elasticity as well as its weak formulation and the corresponding Galerkin problem. The body is assumed to be linearly elastic and the process is assumed to be static. Furthermore we need the following assumptions on the data:
**$H(\mathcal{E})$**: the elasticity operator $\mathcal{E} : \Omega \times S^d \to S^d$ is a bounded symmetric positive definite fourth order tensor, i.e.

$$
\begin{align*}
(a) & \quad \mathcal{E}_{ijkl} \in L^\infty(\Omega), \quad 1 \leq i,j,k,l \leq d, \\
(b) & \quad \mathcal{E}\sigma \cdot \tau = \sigma \cdot \mathcal{E}\tau, \quad \forall \sigma,\tau \in S^d, \text{ a.e. in } \Omega, \\
(c) & \quad \mathcal{E}\tau \cdot \tau \geq m|\tau|^2 \quad \forall \tau \in S^d, \text{ a.e. in } \Omega \text{ with } m > 0.
\end{align*}
$$

**$H(f)$**: the force and the traction densities satisfy

$$
f_0 \in L^2(\Omega)^d, \quad f_2 \in L^2(\Gamma_2)^d,
$$

**$H(\nu)$**: the threshold at which the bonds between the body and foundation break is denoted by $M > 0$ and the force rate of the bonds is denoted by $a > 0$

**Problem $P_M$**: Find a displacement field $u : \Omega \to \mathbb{R}^d$ and a stress field $\sigma : \Omega \to S^d$ such that

$$
\begin{align*}
\sigma &= \mathcal{E}(\varepsilon(u)) \quad \text{in } \Omega, \\
\text{Div} \sigma + f_0 &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \Gamma_D, \\
\sigma\nu &= f_2 \quad \text{on } \Gamma_N, \\
u_\tau &= 0 \quad \text{on } \Gamma_C, \\
-\sigma\nu &= \begin{cases} 
[0, \infty) & \text{if } u_\nu = 0 \\
\{au_\nu\} & \text{if } u_\nu \in (-M, 0) \\
[0, -aM] & \text{if } u_\nu = -M \\
\{0\} & \text{if } u_\nu < -M
\end{cases} \quad \text{on } \Gamma_C.
\end{align*}
$$

Condition (7) means that there is no tangential displacement on the contact boundary. It is easy to observe that in real situation the elastic body could displace tangentially on the boundary under the influence of external and mass load. We see that the condition (7) is simplification, but in some cases, especially when displacements are small it could be a good approximation of real situation.

The normal contact multivalued law (8) means that there is no penetration between the body and the foundation while the law (9) means that if there is contact (i.e. $u_\nu = 0$) then the reaction force assumes some nonnegative value, and if there is no contact the bonding force is proportional to the displacement below the bonding threshold and equal to zero above the bonding threshold.

In order to formulate the laws (8) and (9) in the inclusion form, we introduce
the following functions

\[
    j_1(s) = \begin{cases} 
    \frac{aM^2}{s^2} & \text{if } s < -M \\
    \frac{a^2}{s^2} & \text{if } s \in [-M, 0] \\
    0 & \text{if } s > 0.
    \end{cases}
\]

(10)

\[
    j_2(s) = \begin{cases} 
    0 & \text{if } s \leq 0 \\
    +\infty & \text{if } s > 0.
    \end{cases}
\]

(11)

Observe that this laws cannot be written by means of Clarke subdifferential only (since the notion of Clarke subdifferential requires the function to be locally Lipschitz and in presented case \(j_2\) assumes infinite values). The functional \(j_1\) is locally Lipschitz and \(j_2\) is convex. Moreover (8)-(9) is equivalent to say that

\[\sigma_\nu \in \partial_{\text{CL}} j_1(u_\nu) + \partial_{\text{Conv}} j_2(u_\nu)\]

Fig. 1. The plot of the multivalued law used in analyzed model.

An important property of \(j_1\) used in the sequel will be

\[H(j_1): \text{ for all } r \in \mathbb{R} \text{ and for all } \eta \in \partial_{\text{CL}} j_1(u_\nu) \text{ we have } |\eta| \leq K \text{ with a positive real constant } K.\]

3.1. Weak formulation and Galerkin method

We define the space

\[V = \{ v \in H_1 : v = 0 \text{ on } \Gamma_D, v_\tau = 0 \text{ on } \Gamma_C \}.\]

Moreover we define an operator \(A : V \to V^*\) by

\[
\langle Au, v \rangle = \int_\Omega \varepsilon_{ij} \varepsilon_{ij}(u) \varepsilon_{kl}(v) \, dx
\]
and a functional $f \in V^*$ by

$$\langle f, v \rangle = \int_\Omega f_0 \cdot v \, dx + \int_{\Gamma_N} f_2 \cdot v \, d\Gamma.$$  

The norm in the space $V$ is defined as $\|v\|_V^2 = \langle Av, v \rangle$. This expression defines the norm equivalent to the Sobolev norm $H^1(\Omega)^d$ due to Korn inequality and $H(E)$.

Furthermore we denote by $K \subset V$ the cone

$$K = \{ v \in V : v_\nu \leq 0 \text{ on } \Gamma_C \}.$$  

Now the weak formulation of our problem will be the following

**Problem $P_V$.** Find a displacement field $u \in K$ and $\xi \in L^2(\Gamma_C)$ such that $\xi \in \partial_{C1j_1}(u_\nu)$ a.e. in $\Gamma_C$ and for every $v \in K$ we have

$$\langle Au - f, v - u \rangle + \int_{\Gamma_C} \xi(v_\nu - u_\nu) \, d\Gamma \geq 0. \quad (12)$$  

Existence of solutions to above problem is well known (see [6],[7]). For the sake of numerical solution of above problem we define the sequence of finite element spaces $\{V_n\}_{n=1}^\infty$ which are finite dimensional and approximate $V$ from inside, i.e. $\text{el}(\bigcup_{n=1}^\infty V_n) = V$. We denote $K_n = V_n \cap K$ and moreover we assume that $\text{el}(\bigcup_{n=1}^\infty K_n) = K$. The Galerkin problem will be defined as follows

**Problem $P_{V_n}$.** Find a displacement field $u^n \in K_n$ and $\xi^n \in L^2(\Gamma_C)$ such that $\xi^n \in \partial_{C1j_1}(u^n_\nu)$ a.e. in $\Gamma_C$ and for every $v^n \in K_n$ we have

$$\langle Au^n - f, v^n - u^n \rangle + \int_{\Gamma_C} \xi^n(v^n_\nu - u^n_\nu) \, d\Gamma \geq 0. \quad (13)$$  

**Theorem 1.** Under assumptions $H(E), H(f), H(\nu)$ if $(u^n, \xi^n)$ is a sequence of solutions to $P_{V_n}$ then, for a subsequence we have $u^n \rightharpoonup u$ weakly in $V$ and $\xi^n \rightharpoonup \xi$ weakly in $L^2(\Gamma_C)$ where $u$ and $\xi$ solves $P_V$.

For the proof of Theorem 1 see [4]

**3.2. The minimization problem**

The numerical method will consist in solving the following minimization problem

**Problem $M_{V_n}$.** Find a displacement field $u^n \in K_n$ such that for the functional

$$J_n(v) = \frac{1}{2} \langle Av, v \rangle - \langle f, v \rangle + \int_{\Gamma_C} j_1(v_\nu(x)) \, d\Gamma$$
we have $J_n(u^n) = \min_{v \in K_n} J_n(v)$.

**Theorem 2.** For every $n$ the functional $J_n$ has a global minimum over $K_n$. Moreover this minimum is a solution of $\mathcal{P}_{\nu_n}$.

For the proof of Theorem 2 see [4].

Let us assume that $\{v_1, \ldots, v_n\}$ is a base of $V_n$. We define $\Pi_n : V_n \rightarrow \mathbb{R}^n$ as the mapping which associates to the function $v \in V_n$ its coordinates in this base. Now $K_n$ will be the cone defined as $\Pi_n(K_n)$. The problem $\mathcal{M}_{\nu_n}$ can be equivalently reformulated as follows

**Problem $\mathcal{M}_{\mathbb{R}^n}$:** Find $x = (x_1, \ldots, x_n) \in K_n$ such that for the functional

$$J_n(x) = \frac{1}{2} z^T A z - F^T z + \int_{\Gamma_C} j_1 \left( \sum_{i=1}^n z_i v_i \right) \nu_1(z) \, d\Gamma$$

we have $J_n(x) = \min_{z \in K_n} J_n(z)$.

In the above problem the matrix $A$ is defined as $A = \{\langle Av_i, v_j \rangle\}_{i,j=1}^n$ and the vector $F$ as $F = \{\langle f, v_i \rangle\}_{i=1}^n$.

Obviously $u_n$ solves $\mathcal{M}_{\nu_n}$ if and only if $\Pi_n(u_n)$ solves $\mathcal{M}_{\mathbb{R}^n}$. Numerically we will solve this problem using Proximal Bundle Method.

## 4. Proximal Bundle Method

Proximal Bundle Method is used for nonsmooth and nonconvex optimization of locally Lipshitz functionals. It can be used as a blackbox tool to minimize a functional $J_n : K_n \rightarrow \mathbb{R}$ provided for any $x \in K_n$ we can compute $J_n(x)$ and we can find (one of possibly many) $\eta \in \partial \text{Cl} J_n(x)$.

The method is well known (see [13],[14]) and has been used in context of hemivariational inequalities (see [7]) however it uses many parameters which are somewhat arbitrary in choice and no sensitivity analysis of the method with respect to the parameter choice in application to hemivariational inequalities is known. This article aims to fill this gap.

The main idea of the method is the construction of two sequences $x_i, y_i \in \mathbb{R}^n$. The points $x_i$ will converge to the minimum of $J_n$ and $y_i$ will be the auxiliary points at which the subdifferentials will be computed and accumulated during the course of the algorithm. The elements of $\partial \text{Cl} J_n(y_i)$ which will be computed during the algorithm will be denoted by $\eta_i$. The starting point for the iteration can be arbitrary $x_1 = y_1 = 0$.

Each iteration step consists of two phases:

1. **Direction finding.**
2. **Line search.**
**Direction finding** In the $k$-th iteration step we have two sequences of points $x_j, y_j$ and set of subgradients $\eta_j$, where $j \in \{1, \ldots, k\}$. Main aim of direction finding phase is to find the direction $d_k \in \mathbb{R}^n$. The idea is to approximate functional $J_n$ by piecewise linear function $\hat{J}_n^k$ defined below

$$
\hat{J}_n^k(x) = \max_{j \in \{1, \ldots, k\}} \{ J_n(y_j) + \eta_j^T (x - y_j) \},
$$

which can be written by

$$
\hat{J}_n^k(x) = \max_{j \in \{1, \ldots, k\}} \{ J_n(x_k) + \eta_j^T (x - x_k) - \alpha_{kj} \},
$$

where $\alpha_{kj}$ is the linearization error defined by

$$
\alpha_{kj} = J_n(x_k) - J_n(y_j) - \eta_j^T (x_k - y_j) \text{ for all } j \in \{1, \ldots, k\}.
$$

If the functional $J_n$ is convex, then it is easy to show that $\hat{J}_n^k(x) \leq J_n(x)$ for all $x \in \mathbb{R}^n$ and $\alpha_{kj} \geq 0$ for all $j \in \{1, \ldots, k\}$ (see [14]). In the nonconvex case $\hat{J}_n^k(x)$ could be greater than $J_n(x)$ and $\alpha_{kj}$ could be less than zero. Its the reason why $\alpha_{kj}$ was replaced by subgradient locality measure (see [13]). It is

$$
\beta_{kj} = \max\{|\alpha_{kj}|, \gamma(s_j^k)^2\},
$$

where $\gamma \geq 0$ is the distance measure parameter and $s_j^k$ is defined by

$$
s_j^k = \|x_j - y_j\| + \sum_{i=j}^{k-1} \|x_{i+1} - x_i\|
$$

is distance measure. Note that $\beta_{kj} \geq 0$ for all $j \in \{1, \ldots, k\}$.

To calculate the search direction $d_k$ we replace original problem by the cutting plane problem (see [13],[14])

**Problem CP**

$$
\begin{cases}
\text{minimize} & \hat{J}_n^k(x_k + d) + \frac{1}{2} u_k d^T d \\
\text{subject to} & x_k + d \in \mathbb{K}_n
\end{cases}
$$

where $\frac{1}{2} u_k d^T d$ is the regularizing quadratic penalty term. This regularization is needed to guarantee the existence of the solution $d_k$. Parameter $u_k$ is added to improve the convergence rate and to accumulate some additional information about the curvature of $J_n$ around $x_k$ (see [13],[14]).

Problem CP is still nonsmooth optimization problem but it can be rewritten as a smooth quadratic programming problem.

**Problem QP** Find the solution $(d_k, v_k) \in \mathbb{R}^{n+1}$ of

$$
\begin{cases}
\text{minimize} & v + \frac{1}{2} u_k d^T d \\
\text{subject to} & -\beta_{kj} + \eta_j^T d \leq v \text{ for all } j \in \{1, \ldots, k\} \text{ and } x_k + d \in \mathbb{K}_n
\end{cases}
$$
This problem is well known and can be solved by standard convex quadratic programming algorithms [13]. In the computations we used the library for quadratic programming, which is based on the method proposed in [5]. This method is based on the Sequential Minimal Optimization algorithm with an improved working set selection strategy.

**Line search** In this phase we are looking for the most appropriate values for $x_{k+1}$ and $y_{k+1}$. Notice that $d_k$ calculated in previous section minimizes only the approximation $\hat{J}_n^k$, so $x_{k+1} = x_k + d_k$ is not necessarily the best possible value. We know that $d_k$ is good direction, now we consider the problem of determining how long should be the step size into that direction. We assume that $m_L \in (0, \frac{1}{2})$, $m_R \in (m_L, 1)$ and $t \in (0, 1]$ are fixed method parameters. Firstly we search for the largest number $t^k_L \in [0, 1]$ such that

$$J_n(x_k + t^k_L d_k) \leq J_n(x_k) + m_L t^k_L v_k,$$

where $v_k$ is the descent ratio and $v_k = \hat{J}_n^k(x_k + d_k) - \hat{J}_n(x_k) < 0$. If such a parameter $t^k_L$ exists we take a long step

$$x_{k+1} = x_k + t^k_L d_k, \quad y_{k+1} = x_{k+1}.$$

Otherwise, if (14) holds but $0 < t^k_L < t$ then a short step

$$x_{k+1} = x_k + t^k_L d_k, \quad y_{k+1} = x_k + t^k_R d_k$$

is taken, where $t^k_R > t^k_L$ and

$$-\beta_{k+1}^{k+1} + \eta^T d_k \geq m_R v_k.$$  

(15)

If $t^k_L = 0$ then we take null step

$$x_{k+1} = x_k, \quad y_{k+1} = x_k + t^k_R d_k.$$

In long step we have a significant decrease in the value of the functional $J_n$ and there is no need for detecting discontinuities in the gradient of $J_n$. In short step and null step there exists discontinuity in the gradient of $J_n$. Then the Formula (15) ensures that $x_k$ and $y_{k+1}$ lie on the opposite sides of this discontinuity, and new subgradient will force important modification on the next direction finding phase.

In the computations a simple bisection algorithm is used for finding $t^k_L$ and $t^k_R$. In both cases we take the biggest possible values.

**Stop criterion** The iteration is terminated if $v_k \geq -\varepsilon_s$ where $\varepsilon_s > 0$ is a final accuracy tolerance supplied by the user (see [13],[14]).

The method has five parameters $(u_k, \gamma, \bar{t}, m_L, m_R)$. In the sequel we analyze the convergence speed with respect to those parameters.
5. Problem Setup

The computations were performed on an academic two dimensional example. The physical setting is shown in Figure 2.

We assume that $\Omega = (0, L_1) \times (0, L_2) \subset \mathbb{R}^2$ and the boundary is divided into regions $\Gamma_D = \{0\} \times (0, L_2) \cup \{L_1\} \times (0, L_2)$, $\Gamma_N = (0, L_1) \times \{L_2\}$ and $\Gamma_C = (0, L_1) \times \{0\}$. The two dimensional example can be treated as an approximation for a three dimensional case if domain $\Omega$ is the cross-section of a three dimensional linear elastic body. In our case the body is assumed to be thin such that the plane stress hypothesis can be assumed. We furthermore assume that the body is isotropic and homogeneous so that the constitutive tensor can be described using only two parameters: the Young modulus $E$ and the Poisson ratio $\nu$. The exact form of the constitutive tensor used in the computations is the following

$$
(\mathcal{E} \tau)_{\alpha \beta} = \frac{E \nu}{1 - \nu^2} (\tau_{11} + \tau_{22}) \delta_{\alpha \beta} + \frac{E}{1 + \nu} \tau_{\alpha \beta}, \quad 1 \leq \alpha, \beta \leq 2,
$$

where $\delta_{\alpha \beta}$ is the Kronecker symbol.

The following data was used in the computations

$L_1 = 40 \text{ cm}, \ L_2 = 4 \text{ cm}, \ \nu = 0.33, \ E = 69 \text{ GPa}.$

The values $\nu, E$ correspond to the physical properties of aluminium. The forces were assumed to be the following

$$
f_0 \equiv (0, 0), \quad f_2 = \begin{cases} (0, 0.69 \text{ GPa}) \text{ on } (0, L_1/2) \times \{L_2\}, \\ (0, -0.345 \text{ GPa}) \text{ on } (L_1/2, L_1) \times \{L_2\}. \end{cases}
$$
The values used in the contact law chosen for the computations were
\[ M = 2 \text{ mm}, a = 69 \text{ GPa/m}. \]

In order to avoid the numerical locking effects we used the structural crossed mesh of triangles as presented in Figure 3.

![Fig. 3. The example mesh used in the computations for the test problem.](image)

The deformed mesh for the after the computations for 710 degrees of freedom (mesh 40 × 4) is presented in Figure 4. The normal stress and displacement on the contact boundary are presented in Figure 5.

![Fig. 4. The deformed mesh obtained as the computation result for the test problem.](image)

The verification of the method correctness is done via calculation of the residual and comparing it to the deformation obtained from the contact law. The computed total normal force and displacement values versus the ideal values from the contact boundary condition are presented in Figure 6.

6. Tests of sensitivity

The convergence speed of the Proximal Bundle Method was tested for various values of its parameters. According to [13], [14] the key parameter is \( u_k \). In the sequel we present the analysis of convergence speed for various values of \( u_k \) and other parameters. The values of \( y_k \) in all the simulations were accumulated for 200 steps.
Fig. 5. The total normal force on every element and displacement on the contact boundary obtained for the test problem. The values are divided 4 times for visibility.

of the iteration. After every 200 steps the stop criterion was checked and if it did not hold the points $y_k$ were deleted from memory and the procedure was repeated starting from obtained $x_k$.

6.1. Dependance of convergence speed on $u_k$ for the mesh $40 \times 4$.

In this section the dependance of the convergence speed of PBM on the choice of $u_k$ for the mesh $40 \times 4$ (710 degrees of freedom) is analyzed. The minimal obtained values of the functional are summarized in the Table 1. Moreover in this table there is presented the total number of iteration steps until the stop criterion was obtained as well as the number of long, short and null steps respectively. The other parameters in this and the following simulations were: $\gamma = 0.5, m_L = 0.001, m_R = 0.5, \ell = 0.003, \varepsilon_s = 10^{-13}$. The starting point for all the cases was the function constantly equal to 0.

Clearly for too small and too large $u_k$ the convergence occurs after large number of steps comparing with the intermediate values of $u_k$. Moreover for large values of $u_k$ we have mostly null steps, while for too small values we have mainly short steps. For the optimal $u_k$ values (as it is clear from the Table 1 these are values from 0.0007 to 0.001) we have mostly short steps and some null steps. The fastest convergence rate equal to $-14$, i.e. $\mathcal{J}(x_k) - \mathcal{J}_{\text{min}} \approx Ck^{-14}$ was obtained for $u_k = 0.0008$. The rate close to $-3$ (in accordance with [9]) was obtained for $u_k = 0.0005$. For values $u_k = 5$ and $u_k = 0.5$ during the iteration history the discontinuity which was hard to pass was encountered and the objective did not improve significantly for many iterations as it is seen in the logarithmic graph.
Fig. 6. The theoretical graph of total normal force vs the displacement on the contact boundary (continuous line) and the respective values obtained for the points on the boundary (dots).

6.2. Dependance of convergence speed on $u_k$ for the mesh $80 \times 8$.

Now we present the analysis of the convergence speed of PBM for the mesh $80 \times 8$ (2702 degrees of freedom). Since the Galerkin space $V_n$ of admissible solutions is larger, the optimal objective value is smaller now. As it is clear from the Table 2 and Figure 8 the optimal $u_k$ value is 0.0002 which corresponds to the value for the mesh $40 \times 4$ divided by 4. Moreover the optimal convergence rate is again about $-14$. We can hypothesise that $u_k \sim h^2$, where $h$ is the mesh size parameter.

6.3. Dependance of convergence speed on $u_k$ for the mesh $160 \times 16$.

The problem considered in this subsection has 10526 degrees of freedom. If the hypothesis that $u_k$ should be proportional to $h^2$ is right, then the optimal value of $u_k$ for this mesh should be about 0.00005. The numerical tests, as it is clear from the Table 3 and Figure 9 indeed show that for this value the convergence was fastest and the optimal objective value was only slightly worse then for $u_k = 0.00002$.

6.4. Test of PBM behavior for changing $u_k$ value

In [13] it is suggested that the adaptive changing of $u_k$ should improve the algorithm efficiency. In order to test this, we took the solution obtained for the mesh $80 \times 8$ for $u_k = 0.0002$ using $\varepsilon_s = 10^{-13}$ and run the optimization algorithm again with
<table>
<thead>
<tr>
<th>$u_k$</th>
<th>iterations</th>
<th>$J_{min}[10^{-4}]$</th>
<th>$J(x_{1400})[10^{-4}]$</th>
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<th>short</th>
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<td>145</td>
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<td>-5.28867208</td>
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<td>0</td>
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</table>

Fig. 7. The convergence history for various $u_k$ values for the mesh $40 \times 4$. The graph (a) shows the iteration number (x axis) vs the objective value (y axis). The same graph with logarithmic scale for both axes is depicted in the graph (b). Selected plots with fitted lines on logarithmic scale are depicted in the graph (c).
Tab. 2. Results of tests for various $u_k$ values for the mesh $80 \times 8$.

<table>
<thead>
<tr>
<th>$u_k$</th>
<th>iterations</th>
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<th>$J(x_{14000}) \times 10^{-4}$</th>
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<th>short</th>
<th>null</th>
</tr>
</thead>
<tbody>
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<td>273</td>
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<td>533</td>
</tr>
<tr>
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<td>$-5.37502537$</td>
<td>0</td>
<td>3294</td>
<td>106</td>
</tr>
<tr>
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<td>$-5.37504377$</td>
<td>$-5.37503723$</td>
<td>0</td>
<td>2656</td>
<td>144</td>
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<tr>
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<td>2800</td>
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<td>$-5.37503989$</td>
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<td>$-5.37504381$</td>
<td>$-5.37504132$</td>
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<td>2716</td>
<td>84</td>
</tr>
<tr>
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</table>

Fig. 8. The convergence history for various $u_k$ values for the mesh $80 \times 8$. The graph (a) shows the iteration number (x axis) vs the objective value (y axis). The same graph with logarithmic scale for both axes is depicted in the graph (b). Selected plots with fitted lines on logarithmic scale are depicted in the graph (c).
**Tab. 3.** Results of tests for various $u_k$ values for the mesh 160 $\times$ 16.

<table>
<thead>
<tr>
<th>$u_k$</th>
<th>iterations</th>
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<th>short</th>
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<td>0</td>
<td>7116</td>
<td>84</td>
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<tr>
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<td>6000</td>
<td>-5.4006639</td>
<td>-5.3920934</td>
<td>0</td>
<td>5953</td>
<td>47</td>
</tr>
<tr>
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<td>5600</td>
<td>-5.4006641</td>
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<td>5546</td>
<td>54</td>
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<td>7800</td>
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<td>40200</td>
<td>0</td>
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</table>

**Fig. 9.** The convergence history for various $u_k$ values for the mesh 160 $\times$ 16. The graph (a) shows the iteration number (x axis) vs the objective value (y axis). The same graph with logarithmic scale for both axes is depicted in the graph (b). Selected plots with fitted lines on logarithmic scale are depicted in the graph (c).
Tab. 4

<table>
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<tr>
<th>$u_k$</th>
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<th>short</th>
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<td>402</td>
<td>20984</td>
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<td>45</td>
<td>755</td>
</tr>
<tr>
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<td>930</td>
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</table>

Fig. 10. The plots showing the decrease of the objective value obtained after the iteration for the mesh $80 \times 8$ for the (best) value $u_k = 0.0002$. The solution was taken as the starting point for the iteration for various values of $u_k$. The plot (a) shows the objective value (y axis) vs number of iterations (x axis) and the same plot with logarithmic scales for both axes is shown in the plot (b).

$\varepsilon_s = 10^{-15}$ and various values of $u_k$ taking the obtained solution as the starting point. The results, presented in Figure 10 and Table 4, show, that the objective did not decrease significantly, so indeed the minimal value was found previously. The slight improvement of the minimal value was possible for the large values of $u_k$.

6.5. The dependence of convergence speed on $\bar{t}$

Now we present the results of the tests of dependance of the convergence speed on the parameter $\bar{t}$. In this and the following sections the mesh $40 \times 4$ was considered and the value of $u_k$ was assumed to be 0.00085. The results are depicted in Figure 11 and Table 5. Clearly, the convergence speed and the optimal objective value is almost independent on the choice of $\bar{t}$. This parameter, however, influences the rate between the long and short steps. For small values of $\bar{t}$ almost all steps are long while for larger values we have almost only short steps.
Tab. 5. Results of tests for various $\bar{t}$ values for the mesh $40 \times 4$.

<table>
<thead>
<tr>
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<th>short</th>
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<td>1255</td>
<td>145</td>
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<tr>
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</table>

Fig. 11. The behavior of algorithm for the mesh $40 \times 4$ for various values of $\bar{t}$. The plot (a) presents the objective value vs the number of the iteration step. The same plots in the logarithmic scale are depicted in the plot (b). The plot (c) shows the relation between the number of the short and long steps.
Tab. 6. Results of tests for various $\gamma$ values for the mesh $40 \times 4$.

<table>
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Fig. 12. The behavior of algorithm for the mesh $40 \times 4$ for various values of $\gamma$. The plot (a) presents the objective value vs the number of the iteration step. The same plots in the logarithmic scale are depicted in the plot (b).

6.6. The dependence of convergence speed on $\gamma$

This section is devoted to the results of the tests of dependance of the convergence speed on the parameter $\gamma$. The results are depicted in Figure 12 and Table 6. Clearly, for too small values of $\gamma$ the algotithm behaves poorly. Above the certain threshold the convergence speed and the optimal objective value is almost independent on the choice of $\gamma$, with the best results obtained for values equal to about 0.9.

6.7. The dependence of convergence speed on $m_L$ and $m_R$

Now we present the results of the tests of dependance of the convergence speed on the parameter $m_L$ and $m_R$. The results for $m_L$ are depicted in Figure 13 and Table 7 and the results for $m_R$ are depicted in Figure 14 and Table 8. Clearly, the convergence speed of the algorithm is not sensitive on the choice of these two parameters.
Tab. 7. Results of tests for various $m_L$ values for the mesh $40 \times 4$.

<table>
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Fig. 13. The behavior of algorithm for the mesh $40 \times 4$ for various values of $m_L$. The plot (a) presents the objective value vs the number of the iteration step. The same plots in the logarithmic scale are depicted in the plot (b).

Tab. 8. Results of tests for various $m_R$ values for the mesh $40 \times 4$.

<table>
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<td>0</td>
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<td>145</td>
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</table>
Fig. 14. The behavior of algorithm for the mesh $40 \times 4$ for various values of $m_R$. The plot (a) presents the objective value vs the number of the iteration step. The same plots in the logarithmic scale are depicted in the plot (b).

7. References


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