VECTOR BUNDLES ON REAL ALGEBRAIC CURVES

by Łukasz Maciejewski

Abstract. We prove that any topological real line bundle on a compact real algebraic curve $X$ is isomorphic to an algebraic line bundle. The result is then generalized to vector bundles of an arbitrary constant rank. As a consequence we prove that any continuous map from $X$ into a real Grassmannian can be approximated by regular maps.

1. Introduction. Throughout this paper $X$ denotes a compact real algebraic curve, that is, a compact 1-dimensional algebraic subset of $\mathbb{R}^d$ for some $d \in \mathbb{N}$. We refer to [1] for terminology and background material on real algebraic geometry. In this paper all vector bundles are real vector bundles. Recall that algebraic vector bundles on $X$ correspond to finitely generated projective modules over the ring of real-valued regular functions on $X$, cf. [1] p. 302. Our main goal is the following:

Theorem 1.1. Any topological line bundle on $X$ is isomorphic to an algebraic line bundle.

Theorem 1.1 is proved in section 2. It can be easily generalized.

Corollary 1.2. Any topological constant rank vector bundle on $X$ is isomorphic to an algebraic vector bundle.

Proof. Any topological vector bundle on $X$ of constant rank $r \geq 1$ splits off a trivial vector bundle of rank $r - 1$, since $\dim(X) = 1$. Hence it suffices to apply Theorem 1.1.

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As a consequence of Corollary 1.2, we obtain a counterpart of the classical Weierstrass approximation theorem for maps from $X$ into the Grassmann variety $G_{n,k}$ of $k$-dimensional vector subspaces of $\mathbb{R}^n$.

**Corollary 1.3.** Let $f : X \to G_{n,k}$ be a continuous map. Each neighborhood of $f$ in the compact-open topology contains a regular map.

**Proof.** It suffices to show that the pullback vector bundle $f^*\gamma_{n,k}$ on $X$, where $\gamma_{n,k}$ is the tautological vector bundle on $G_{n,k}$, is isomorphic to an algebraic vector bundle, cf. [1, Theorem 13.3.1]. This however follows from Corollary 1.2. $\square$

Since the real variety $G_{2,1}$ is biregularly isomorphic to the unit circle

$$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$

we immediately get:

**Corollary 1.4.** Let $f : X \to S^1$ be a continuous map. Each neighborhood of $f$ in the compact-open topology contains a regular map.

All the results above are proved in [1] under the assumption that the curve $X$ is nonsingular. The arguments presented in [1] do not directly generalize to yield Theorem 1.1.

**Corollary 1.5.** For every cohomology class $u$ in $H^1(X; \mathbb{Z}/2)$, there exists a regular map $f : X \to S^1$ such that $f^*(s_1) = u$, where $s_1$ is the unique generator of the cohomology group $H^1(S^1; \mathbb{Z}/2) \cong \mathbb{Z}/2$.

**Proof.** There is a one-to-one correspondence between the homotopy classes of continuous maps from $X$ into $S^1$ and the cohomology classes in $H^1(X; \mathbb{Z})$, cf., [2, p. 300]. Since the reduction modulo 2 homomorphism $H^1(X; \mathbb{Z}) \to H^1(X; \mathbb{Z}/2)$ is surjective, it follows that each cohomology class in $H^1(X; \mathbb{Z}/2)$ is of the form $f^*(s_1)$ for some continuous map $f : X \to S^1$. According to Corollary 1.4, the map $f$ can be assumed to be regular. $\square$

Let us note that Corollary 1.5 implies Theorem 1.1. Indeed, let $\xi$ be a topological line bundle on $X$. The first Stiefel-Whitney class $w_1(\gamma_{2,1})$ of the tautological line bundle $\gamma_{2,1}$ on $G_{2,1}$ generates the cohomology group $H^1(G_{2,1}; \mathbb{Z}/2)$. According to Corollary 1.5, there exists a regular map $f : X \to G_{2,1}$ satisfying $w_1(\xi) = f^*(w_1(\gamma_{2,1})) = w_1(f^*\gamma_{2,1})$. Since topological line bundles are classified by the first Stiefel-Whitney class (cf. [3, Proposition 3.10]), it follows that $\xi$ is isomorphic to the algebraic line bundle $f^*\gamma_{2,1}$. However, we do not know how to prove Corollary 1.5 without making use of Theorem 1.1.
2. Line bundles on real algebraic curves. We first recall a useful construction of algebraic line bundles on an arbitrary affine real algebraic variety $V$. Lemma 2.1 below is a special case of [1] Theorem 12.1.11.

Lemma 2.1. Let $\{U_1, \ldots, U_r\}$ be a Zariski open cover of $V$ and let $h_{ij} : U_j \longrightarrow \mathbb{R}$ be a regular function satisfying $h_{ij}(U_i \cap U_j) \subset \mathbb{R} \setminus \{0\}$ for $1 \leq i, j \leq r$. Assume that $h_{ij} \cdot h_{jk} = h_{ik}$ on $U_j \cap U_k$ for all $i, j, k$, and $h_{ii}(x) = 1$ for all $i$ and $x$ in $U_i$. Let

$E = \{(x, (v_1, \ldots, v_r)) \in V \times \mathbb{R}^r : v_i = h_{ij}(x)v_j \text{ for } x \in U_j, 1 \leq i, j \leq r\}$

and let $p : E \longrightarrow V$ be defined by $p(x, (v_1, \ldots, v_r)) = x$. Then $\xi = (E, p, V)$ is an algebraic line subbundle of the product vector bundle on $V$ with total space $V \times \mathbb{R}^r$, and the map

$U_i \times \mathbb{R} \longrightarrow p^{-1}(U_i), (x, v) \mapsto (h_{i1}(x)v, \ldots, h_{ir}(x)v)$

is an algebraic trivialization of $\xi$ over $U_i$ for $1 \leq i \leq r$.

For any vector bundle $\eta$ and any global section $s$ of $\eta$, let $Z(s)$ denote the zero locus of $s$.

The set $\text{Reg}(X)$ of nonsingular points of $X$ in dimension 1 is a Zariski open subset of $X$, cf. [1] p. 69]. Furthermore, $\text{Reg}(X)$ is a 1-dimensional $C^\infty$ manifold.

Lemma 2.2. Let $x_0$ be a point in $\text{Reg}(X)$. There exists an algebraic line bundle $\xi = (E, p, X)$ on $X$ which admits an algebraic section $s : X \longrightarrow E$ such that $Z(s) = \{x_0\}$ and the restriction of $s$ to $\text{Reg}(X)$ is transverse to the zero section of $\xi$.

Proof. Let $\mathcal{R}_X$ be the sheaf of real-valued regular functions on $X$. For any point $x$ on $X$, we identify the stalk $\mathcal{R}_{X,x}$ with the localization of the ring $\mathcal{R}_X(x)$ at the maximal ideal

$m_x = \{f \in \mathcal{R}_X(x) : f(x) = 0\},$

cf. [1] Proposition 3.2.3). Since the point $x_0$ is in $\text{Reg}(X)$, the stalk $\mathcal{R}_{X,x_0}$ is a regular local ring of dimension 1 and thus a principal ideal domain. In particular, the ideal $m_{x_0} \mathcal{R}_{X,x_0}$ of the ring $\mathcal{R}_{X,x_0}$ is principal. Thus we can find a regular function $f_1$ in $m_{x_0}$ and a Zariski open neighborhood $U_1$ of $x_0$ in $\text{Reg}(X)$ such that

$m_{x_0} \mathcal{R}_X(U_1) = (f_1) \mathcal{R}_X(U_1).$

In particular, $f_1|_{U_1} : U_1 \longrightarrow \mathbb{R}$ is a $C^\infty$ function for which 0 in $\mathbb{R}$ is a regular value and $(f_1|_{U_1})^{-1}(0) = \{x_0\}$.

Let $f_2$ be any regular function in $m_{x_0}$ with $f_2^{-1}(0) = \{x_0\}$, e.g., a polynomial given by the formula $\|x - x_0\|^2$, where $\| \cdot \|$ denotes the euclidean metric
in $\mathbb{R}^d$. We have

$$f_2|_{U_1} = h_{21}f_1|_{U_1}$$

for some regular function $h_{21} : U_1 \to \mathbb{R}$. If $U_2 = X \setminus \{x_0\}$, then

$$h_{12} = \frac{f_1}{f_2} : U_2 \to \mathbb{R}$$

is a regular function on $U_2$. By construction, the sets $h_{21}(U_1 \cap U_2)$ and $h_{12}(U_1 \cap U_2)$ are contained in $\mathbb{R} \setminus \{0\}$. Define $h_{11} : U_1 \to \mathbb{R}$ and $h_{22} : U_2 \to \mathbb{R}$ to be constant functions identically equal to 1. Let $\xi = (E, p, X)$ be the algebraic line bundle on $X$ determined, as in Lemma 2.1, by the Zariski open cover $\{U_1, U_2\}$ of $X$ and the regular functions $h_{ij}$. Note that

$$s : X \to E, \; s(x) = (x, (f_1(x), f_2(x)))$$

is an algebraic section of $\xi$ with $Z(s) = \{x_0\}$. On the set $U_1$, the section $s$ is represented by the map

$$U_1 \to U_1 \times \mathbb{R}, \; x \mapsto (x, f_1(x)),$$

and hence the restriction of $s$ to $\text{Reg}(X)$ is transverse to the zero section of $\xi$. \hfill \Box

We will now give a convenient description of the first cohomology group $H^1(X; \mathbb{Z}/2)$ of the curve $X$. The subset $X \setminus \text{Reg}(X)$ of $X$ is finite. If $X$ has nonsingular connected components, we choose one arbitrary point in each of those and denote the set of such points by $Z$. The curve $X$ can be regarded as a graph (1-dimensional CW complex) with $(X \setminus \text{Reg}(X)) \cup Z$ as the set of vertices. This assertion is a straightforward consequence of the triangulation theorem for semi-algebraic sets, cf. [1, Theorem 9.2.1].

**Lemma 2.3.** There exist subgraphs $X_1, \ldots, X_n$ of $X$ such that each $X_i$ is homeomorphic to the unit circle $S^1$, and the inclusion maps $X_i \to X$ induce an isomorphism

$$\varphi : H^1(X; \mathbb{Z}/2) \to \bigoplus_{i=1}^n H^1(X_i; \mathbb{Z}/2)$$

**Proof.** Let $K$ be a connected 1-dimensional component of $X$ and let $T$ be a maximal tree of the graph $K$. The quotient map $q : K \to K/T$ is a homotopy equivalence and the quotient space $K/T$ is homeomorphic to the wedge sum of a finite number of pointed circles, [2, p. 153]. Each such pointed circle corresponds to a subset of $K/T$ of the form $q(C)$, where $C$ is a subgraph of $K$ homeomorphic to the unit circle. The inclusion maps $q(C) \to K/T$ induce an isomorphism

$$\psi : H^1(K/T; \mathbb{Z}/2) \to \bigoplus_{C} H^1(q(C); \mathbb{Z}/2)$$
If \( q_C : C \longrightarrow q(C) \) is the restriction of the map \( q \), then the homomorphism
\[
\alpha = \bigoplus_C q_C^* : \bigoplus_C H^1(q(C); \mathbb{Z}/2) \longrightarrow \bigoplus_C H^1(C; \mathbb{Z}/2)
\]
is an isomorphism. The homomorphism
\[
q^* : H^1(K/T; \mathbb{Z}/2) \longrightarrow H^1(K; \mathbb{Z}/2)
\]
is an isomorphism, the quotient map being a homotopy equivalence. Finally, the inclusion maps \( C \hookrightarrow K \) induce a homomorphism
\[
\varphi_K : H^1(K; \mathbb{Z}/2) \longrightarrow \bigoplus_C H^1(C; \mathbb{Z}/2)
\]
satisfying \( \varphi_K \circ q^* = \alpha \circ \psi \). Consequently, \( \varphi_K \) is an isomorphism.

The assertion of the lemma follows, because \( X \) has finitely many connected components.

\[ \square \]

**Proof of Theorem 1.1.** The isomorphism classes of topological line bundles on \( X \) form a group, denoted \( \text{Vect}^1(X) \), with tensor product as the group operation. The first Stiefel–Whitney class gives a group isomorphism between \( \text{Vect}^1(X) \) and the first cohomology group \( H^1(X; \mathbb{Z}/2) \), cf. [3, Proposition 3.10]. Also, note that the isomorphism classes of algebraic vector bundles form a subgroup of \( \text{Vect}^1(X) \). Hence, in view of Lemma 2.3, it remains to construct for each \( i = 1, \ldots, n \) an algebraic line bundle \( \xi_i \) on \( X \) with \( w_1(\xi_i|_{X_i}) \neq 0 \) and \( w_1(\xi_i|_{X_j}) = 0 \) for all \( j \neq i \) (note that \( H^1(X_i; \mathbb{Z}/2) \cong \mathbb{Z}/2 \)). Such a line bundle \( \xi_i \) can be obtained as follows.

Let \( x_i \) be a point in
\[
(X_i \cap \text{Reg}(X)) \setminus \bigcup_{j \neq i} X_j
\]
and let \( \xi = (E, p, X) \) be an algebraic line bundle on \( X \) as in Lemma 2.2 with \( x_0 = x_i \). There exists an algebraic section \( s : X \longrightarrow E \) such that \( Z(s) = \{ x_i \} \) and the restriction of \( s \) to \( \text{Reg}(X) \) is transverse to the zero section of \( \xi \). It follows that the line bundle \( \xi|_{X_j} \) is trivial and \( w_1(\xi|_{X_j}) = 0 \) for \( j \neq i \).

Suppose for a moment that the line bundle \( \xi|_{X_i} \) is trivial, and let
\[
\theta : p^{-1}(X_i) \longrightarrow X_i \times \mathbb{R}
\]
be a topological trivialization of \( \xi|_{X_i} \). Then \( \theta(s(x)) = (x, f(x)) \) for each \( x \) in \( X_i \), where \( f : X_i \longrightarrow \mathbb{R} \) is a continuous function. By construction, \( f^{-1}(0) = \{ x_i \} \). The function \( f \) does not change sign on \( X_i \setminus \{ x_i \} \), the set \( X_i \setminus \{ x_i \} \) being homeomorphic to \( \mathbb{R} \). This however is impossible since \( s \) is transverse to the zero section of \( \xi \) in a neighborhood of \( x_i \). Consequently, the line bundle \( \xi|_{X_i} \) is nontrivial and \( w_1(\xi|_{X_i}) \neq 0 \).

We complete the proof by setting \( \xi_i = \xi \). \( \square \)
References


http://www.math.cornell.edu/~hatcher/VBKT/VBpage.html

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Institute of Mathematics
Jagiellonian University
Lojasiewicza 6
30-348 Kraków, Poland