MINIMAL SUBVARIETIES OF INVOLUTIVE RESIDUATED LATTICES

Daisuke Souma

ABSTRACT

It is known that classical logic CL is the single maximal consistent logic over intuitionistic logic Int, which is moreover the single one even over the substructural logic FL$_{ew}$. On the other hand, if we consider maximal consistent logics over a weaker logic, there may be uncountably many of them. Since the subvariety lattice of a given variety $V$ of residuated lattices is dually isomorphic to the lattice of logics over the corresponding substructural logic $L(V)$, the number of maximal consistent logics is equal to the number of minimal subvarieties of the subvariety lattice of $V$. Tsinakis and Wille have shown that there exist uncountably many atoms in the subvariety lattice of the variety of involutive residuated lattices. In the present paper, we will show that while there exist uncountably many atoms in the subvariety lattice of the variety of bounded representable involutive residuated lattices with mingle axiom $x^2 \leq x$, only two atoms exist in the subvariety lattice of the variety of bounded representable involutive residuated lattices with the idempotency $x = x^2$. 
Daisuke SOUMA

MINIMAL SUBVARIETIES OF INVOLUTIVE RESIDUATED LATTICES

Abstract. It is known that classical logic $\mathbf{CL}$ is the single maximal consistent logic over intuitionistic logic $\mathbf{Int}$, which is moreover the single one even over the substructural logic $\mathbf{FL}_{ew}$. On the other hand, if we consider maximal consistent logics over a weaker logic, there may be uncountably many of them. Since the subvariety lattice of a given variety $\mathcal{V}$ of residuated lattices is dually isomorphic to the lattice of logics over the corresponding substructural logic $\mathbf{L}(\mathcal{V})$, the number of maximal consistent logics is equal to the number of minimal subvarieties of the subvariety lattice of $\mathcal{V}$. Tsinakis and Wille have shown that there exist uncountably many atoms in the subvariety lattice of the variety of involutive residuated lattices. In the present paper, we will show that while there exist uncountably many atoms in the subvariety lattice of the variety of bounded representable involutive residuated lattices with mingle axiom $x^2 \leq x$, only two atoms exist in the subvariety lattice of the variety of bounded representable involutive residuated lattices with the idempotency $x = x^2$.

$^1$For more information on minimal subvarieties, see Chapter 9 of [2]

Received 26 October 2008
1. Introduction

An algebra $A = \langle A, \land, \lor, \cdot, \backslash, /, 1 \rangle$ is a residuated lattice (RL) if $A$ satisfies the following conditions.

(R1) $\langle A, \land, \lor, 1 \rangle$ is a lattice,

(R2) $\langle A, \cdot, 1 \rangle$ is a monoid with the unit 1,

(R3) for $x, y, z \in A$, $x \cdot y \leq z \iff y \leq x \backslash z \iff x \leq z / y$.

(R3) is called the residuation law.

An $RL A$ is bounded (RL$_\bot$) if it has the greatest element $\top$ and least element $\bot$.

An $RL A$ is involutive (InRL) if it has a constant 0, called involution constant, which satisfies the following conditions:

1. $x \backslash 0 = 0 / x$,

2. $0 / (x \backslash 0) = (0 / x) \backslash 0 = x$.

In InRL let us define a unary operation $'$ by $x' = x \backslash 0$. We call $'$ the involution.

An $RL A$ is representable (RRL) if it can be represented as a subdirect product of totally ordered RLs.

A non-trivial algebra $A$ is strictly simple, if it has neither non-trivial proper subalgebras nor non-trivial congruences. Note that the notion of proper subalgebras of an infinite algebra $A$ is given as follows: A subalgebra $B$ of $A$ is proper if $B$ is not isomorphic to $A$. The fact that an algebra has no non-trivial proper subalgebras is enough to establish strict simplicity for a RL. For, congruences on residuated lattices correspond to convex normal subalgebras.

The bottom element $\bot \in A$, when exists, is nearly term-definable, if there is an $n$-ary term-operation $t(\bar{x})$ such that for any $n$-tuple $\bar{a} \neq (1, \ldots, 1)$ of elements $n$-times of $A$, $t(\bar{a}) = \bot$ holds.

A variety is a class of algebras which is closed under homomorphic images (H), subalgebras (S) and direct products (P). For any algebra $A$, $V(A) = \text{HSP}(A)$ is a variety generated by $A$. Alternatively, it is an equational class, i.e. a class of the form $\text{Mod}\{E_i| i \in I\}$, where each $E_i$ is an equation. A non-trivial variety $\mathcal{V}$ is called minimal if $\mathcal{V}$ has only trivial proper subvariety. We denote the variety of